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Global Dimension in Categories of Diagrams

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Let I be a finite partially ordered set considered as a category in the usual sense, and let $\mathfrak{A}$ be any abelian category. The relation of gl dim $\mathfrak{A}^I$ ($\mathfrak{A}^I$ denotes the category of functors from I to $\mathfrak{A}$ and is usually referred to as the category of diagrams in $\mathfrak{A}$ over I) to gl dim $\mathfrak{A}$ is the subject of a paper by Mitchell [3]. Among the questions he raises in his paper is the possible invariance of gl dim $\mathfrak{A}^I$ — gl dim $\mathfrak{A}$ with respect to $\mathfrak{A}$. This paper gives an example of a partially ordered set I with regard to which gl dim $\mathfrak{A}^I$ — gl dim $\mathfrak{A}$ is dependent upon $\mathfrak{A}$.

As Mitchell points out [2, pp. 229–233], the study of $\mathfrak{A}^I$ is one way to approach some problems in the global dimension of rings. Specifically, suppose E is a semigroup of matrix units, that is, $E \subset M_n$ where

$$M_n = \{e_{ij} \mid i, j = 1,..., n\} \cup \{0\},$$

and where the multiplication is that of matrix units: $e_{ij}e_{pq} = \delta_{jp}e_{iq}$. We may form the semigroup algebra over a field F, $F[E]$. Then we may ask what relation there is between the structure of E and the global dimension of $F[E]$. For example, if $E = M_n$, then the global dimension is zero; some other results along these lines have been obtained by Clark [1].

DEFINITION. $\mathfrak{A}^I$ will denote the category of all functors from the category I to the category $\mathfrak{A}$. Alternatively, each functor may be thought of as a diagram in $\mathfrak{A}$ over I [2, p. 66]. For this reason, a category of functors $\mathfrak{A}^I$ is often referred to as a category of diagrams.

We will use the standard notations and conventions concerning $\mathfrak{A}^I$ that may be found in [2] or [3]. That is, for $D \in \mathfrak{A}^I$ ($D : I \to \mathfrak{A}$), $D(i)$ is denoted D_i; for $i \leqslant j$, the morphism from D_i to D_j is denoted D_{ij}; and if $f : D \to D'$ is a natural transformation, $f(i)$ is denoted f_i.

Let $T_i : \mathfrak{A}^I \to \mathfrak{A}$ be defined by $T_i(D) = D_i$ and $T_i(f) = f_i$ where $f : D \to D'$. A left adjoint for T_i, S_i, is defined by $S_i(A)_j = A$ for $i \leqslant j$ and is 0 otherwise; and $S_i(A)_{kl} = 0$ unless $i \leqslant k \leqslant l$ in which case it is 1_A.

The following result due to Mitchell [2, p. 228] is central to the study of the global dimension of $\mathfrak{A}^I$. Only the easily proved "if" part will be needed.

THEOREM 1. *An object D in $\mathfrak{A}^I$ is projective if and only if $D \cong \oplus_{i \in I} S_i(P_i)$ where each P_i is a projective object in $\mathfrak{A}$.*

The following result was obtained independently by Mitchell [3, p. 347] and the author.

DEFINITION. For each $i \in I$, define a functor $M_i : \mathfrak{A} \to \mathfrak{A}^I$ by $M_i(A)_j = 0$ unless $j = i$ in which case it is A; $M_i(A)_{kl} = 0$ for all $k, l \in I$.

The importance of the M_i's as seen in the following result is that it allows the computation of gl dim $\mathfrak{A}^I$ from the homological dimension of members of $\mathfrak{A}^I$ of a comparatively simple structure. This proof is due to the author.

THEOREM 2. *For any D in $\mathfrak{A}^I$,* h.d. $D \leqslant \sup_{i \in I}$ h.d. $M_i(D_i)$. *Hence*

$$\text{gl dim } \mathfrak{A}^I = \sup\{\text{h.d. } M_i(A) \mid i \in I, A \in \mathfrak{A}\}.$$

Proof. We will induct on the number of nonzero D_i's in D. Certainly there is an i in I such that $D_i \neq 0$ and $D_j = 0$ if $j < i$. We have then the short exact sequence

$$0 \to D' \to D \to M_i(D_i) \to 0,$$

where $f : D \to M_i(D_i)$ is defined so that $f_i = 1_{D_i}$ and $f_j = 0$ for $j \neq i$, and where $D' \to D$ is the kernel of f.

From this exact sequence and an elementary result of homological algebra, we have that h.d. $D \leqslant \max(\text{h.d. } M_i(D_i), \text{h.d. } D')$. On the other hand, since $D_j' = D_j$ for $j \neq i$ and $D_i' = 0$, we have by induction that

$$\text{h.d. } D' \leqslant \max_{j \neq i} \text{h.d. } M_j(D_j).$$

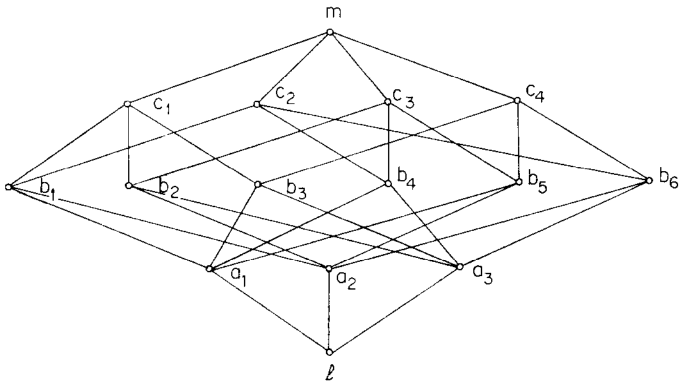
Combination of these two results yields the desired inequality.

The following definition and result are from [3, pp. 345–346].

DEFINITION. The *rank* of an element of a partially ordered set is defined inductively. Minimal elements have rank 0. An element i has rank $\geqslant r$ if and only if it is preceded by two incompatible elements of rank $\geqslant r - 1$. The rank of i is r if $r \leqslant \text{rank } i \not\geqslant r + 1$.

PROPOSITION 3. *Let m be the maximum rank of an element of I and let* h.d. $D_i \leqslant n$ *for all i. Then* h.d. $D \leqslant m + n$.

Let I be the partially ordered set depicted in this Hasse diagram (directed upward).



THEOREM 4. *Let* $\mathscr{F}_i$ *denote the category of vector spaces over a field of characteristic* i. *Then* gl dim $\mathscr{F}_2^I = 4$ and gl dim $\mathscr{F}_0^I = 3$.

Proof. It follows from Proposition 3 that h.d. $M_i(A) \leqslant 3$ for all $i \neq l$ and for all A in $\mathscr{F}_2$ or $\mathscr{F}_0$. It will now be shown that for nonzero A, A in $\mathscr{F}_2$ implies that h.d. $M_l(A) = 4$, and A in $\mathscr{F}_0$ implies that h.d. $M_l(A) = 3$. This will be done by choosing a projective resolution for $M_l(A)$ without regard for the origin of A and then using Theorem 1.

Let $D^0 = S_l(A)$ which is projective by Theorem 1, and let $d^0 : S_l(A) \to M_l(A)$ be the natural map. Now let K^0 denote ker d^0 and define D^1 in terms of K^0 by letting $D^1 = \bigoplus_{i=1}^3 S_{a_i}(K^0_{a_i})$ which is also projective. $d^1 : D^1 \to K^0$ will be the epimorphism induced by the natural maps $S_{a_i}(K^0_{a_i}) \to K^0$. Then D^2 and $d^2 : D^2 \to K^1 = \ker d^1$ are defined similarly. Let $D^2 = \bigoplus_{i=1}^6 S_{b_i}(K^1_{b_i})$, and let d^2 be the epimorphism induced by the maps $S_{b_i}(K^1_{b_i}) \to K^1$. Let $K^2 = \ker d^2$.

Now it will be shown that if $A \in \mathscr{F}_2$, then K^2 is not projective; and if $A \in \mathscr{F}_0$, then K^2 is projective. First notice that $K_i{}^2 = 0$ for $i \neq c_1 ,..., c_4 , m$, and that each $K^2_{c_i}$ may be regarded as a subobject of $K_m{}^2$. Furthermore, a straightforward, if lengthy, calculation will verify that

$$K^2_{c_1} + K^2_{c_2} + K^2_{c_3} = K^2_{c_1} \oplus K^2_{c_2} \oplus K^2_{c_3}.$$

(This in independent of A's origin.)

Let B denote $(K^2_{c_1} \oplus K^2_{c_2} \oplus K^2_{c_3}) \cap K^2_{c_4}$. It follows by a straightforward calculation that if A is in $\mathscr{F}_2$, then $B \neq 0$; and that if A is in $\mathscr{F}_0$, then $B = 0$. Hence by Theorem 1, K^2 is not projective if A is in $\mathscr{F}_2$ and is projective if A is in $\mathscr{F}_0$. Thus h.d. $M_l(A)$ equals 4 and 3, respectively, and it follows by Theorem 2 that gl dim $\mathscr{F}_2^I = 4$ and gl dim $\mathscr{F}_0^I = 3$.

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